Poisson reduction and branes in Poisson-Sigma models

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Abstract. We analyse the problem of boundary conditions for the Poisson-Sigma model and extend previous results showing that non-coisotropic branes are allowed. We discuss the canonical reduction of a Poisson structure to a submanifold, leading to a Poisson algebra that generalizes Dirac's construction. The phase space of the model on the strip is related to the (generalized) Dirac bracket on the branes through a dual pair structure.

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1 Introduction

Poisson-Sigma models ([13], [10]) are topological field theories whose field content is a bundle map from the tangent bundle of a surface Σ to the cotangent bundle of a Poisson manifold M. Their initial interest was due to the fact that some two-dimensional gauge theories such as pure gravity, WZW models and Yang Mills are particular cases of Poisson-Sigma models (maybe after the addition of a non-topological term expressed in terms of Casimir functions of the Poisson structure on M).

The models gained renewed attention with the appearance of [4], where it was shown that when Σ is a disk the perturbative path integral expansion (with appropriate boundary conditions) reproduces the $*$-product introduced by Kontsevich in [12] that gives the deformation quantization of a Poisson manifold.

More recently, A.S. Cattaneo and G. Felder studied in [5] the possible boundary conditions for the Poisson-Sigma model and concluded that the branes of this model are labeled by the coisotropic submanifolds of M. In this paper we show that more general boundary conditions are allowed and, in fact, we extend their procedure to the case in which the base map of the bundle map restricts on $\partial\Sigma$ to an (almost) arbitrary submanifold of M.

We begin with a review of the basic ideas on Poisson geometry in Section 2

(see [14] for a thorough study). Section 3 comprises some well-known facts about the reduction of a Poisson manifold along with some new results and a somewhat original approach to the subject. In section 4 we give a brief presentation of Poisson Sigma models from the lagrangian point of view.

Section 5 is devoted to the study of general boundary conditions for the Poisson-Sigma model. We show explicitly that coisotropy of the brane is not essential and give the precise boundary conditions that the fields must satisfy in the general case, obtaining the results of [5] as a particular case.

In Section 6 we carry out the hamiltonian study of the model for an open string with our boundary conditions. It turns out that there exist a Poisson and an anti-Poisson map from the phase space to the branes at the endpoints of the string when in the latter ones the (generalized) Dirac bracket obtained by Poisson reduction of M is considered.

Section 7 contains our conclusions as well as the discussion on the quantization of the model and future lines of research.

2 Poisson geometry

Let $(\mathscr{A},\cdot)$ be an associative, commutative algebra (we will omit the symbol $\cdot$ for the commutative product on $\mathscr{A}$ in the following) with unit over the real or complex numbers. $(\mathscr{A},\cdot,\{,\})$ is said to be a *Poisson algebra* if:

(i) $\{,\}$ defines a Lie bracket on $\mathscr{A}$

(ii) Leibniz rule (compatibility of both products) is satisfied, i.e.

$$\{x,yz\} = y\{x,z\} + \{x,y\}z,\ \forall x,y,z \in \mathscr{A}$$

When $\mathscr{A}$ is the algebra of smooth functions on a manifold M the concept of Poisson algebra is equivalent to that of *Poisson manifold*. An m-dimensional Poisson manifold (M,Γ) is a differentiable manifold M equipped with a bivector field Γ that makes the algebra of smooth functions a Poisson algebra when the Poisson bracket of two functions in $C^\infty(M)$ is given by the contraction of Γ:

$$\{f,g\}(p) = \iota(\Gamma_p)(df\wedge dg)_p,\ p\in M$$

Taking local coordinates X^i on M, $\Gamma^{ij}(X) = \{X^i,X^j\}$. The Jacobi identity for the Poisson bracket reads in terms of Γ^{ij}:

$$\Gamma^{ij}\partial_i\Gamma^{kl} + \Gamma^{ik}\partial_i\Gamma^{lj} + \Gamma^{il}\partial_i\Gamma^{jk} = 0$$

where summation over repeated indices is understood.

Define $\Gamma^\sharp : T^*M \to TM$ by

$$\beta(\Gamma^\sharp(\alpha)) = \iota(\Gamma)(\alpha\wedge\beta),\ \alpha,\beta \in T^*M$$

By virtue of Jacobi identity, the image of $\Gamma^\sharp$,

$$\mathrm{Im}(\Gamma^\sharp) := \bigcup_{p\in M} \mathrm{Im}(\Gamma^\sharp_p)$$

is a completely integrable (general) differential distribution and M admits a (generalized) foliation (see [14] for definitions of these concepts). M is foliated into leaves which may have varying dimensions. The Poisson structure can be consistently restricted to a leaf and this restriction defines a non-degenerate Poisson structure on it. That is why we will also refer to the leaves as *symplectic leaves* and to the foliation as the *symplectic foliation* of M. This result comes from a generalization of the classical Frobenius theorem for regular distributions.

An example of a Poisson manifold is obtained by taking $M = g^*$, where g is a Lie algebra. Hence, M is a linear space and the Poisson structure is the so called Kostant-Kirillov Poisson structure that, for the linear functions, is given by the Lie bracket of g, i.e.

$$\{f,g\} = [f,g],\ f,g \in g.$$

The symplectic leaves in this case correspond to the orbits under the coadjoint representation of any connected Lie group G with Lie algebra g and have, in general, varying dimensions (in particular, the origin is always a symplectic leaf).

Another way of defining a Poisson algebra for functions on M is via a presymplectic structure, i.e. a closed two-form $\omega \in \Lambda^2(M)$. In this case the Poisson algebra $\mathscr{A}$ consists of functions that possess a hamiltonian vector field, i.e. those functions $f \in C^\infty(M)$ for which the equation

$$\omega(\mathscr{X},\mathscr{Y}) = \mathscr{Y}(f)$$

has a solution $\mathscr{X} \in X(M)$ for any $\mathscr{Y} \in X(M)$. Given $f_1, f_2 \in \mathscr{A}$ with hamiltonian vector fields $\mathscr{X}_1, \mathscr{X}_2$ respectively, $f_1 f_2$ has the hamiltonian vector field $f_1\mathscr{X}_2 + f_2\mathscr{X}_1$ and then $\mathscr{A}$ is a subalgebra of $C^\infty(M)$ (if and only if ω is symplectic the Poisson algebra induced by it gives M the structure of a Poisson manifold). The Poisson bracket is defined by

$$\{f_1, f_2\} = \omega(\mathscr{X}_1, \mathscr{X}_2).$$

Note that in general the hamiltonian vector field $\mathscr{X}_1$ for $f_1 \in \mathscr{A}$ is not uniquely defined but the ambiguities are in the kernel of ω and then it leads to a well defined Poisson bracket. Due to closedness of ω, $\{f_1, f_2\} \in \mathscr{A}$ and the Jacobi identitiy is satisfied. It is worth mentioning that in this case the center of $\mathscr{A}$ (Casimir functions) is the set of constant functions on M.

Given two Poisson manifolds (M_1,Γ_1), (M_2,Γ_2) and a differentiable map $F : M_1 \to M_2$, F is a Poisson map if

$$\{f,g\}_2 \circ F = \{f\circ F, g\circ F\}_1,\ \forall f,g \in C^\infty(M_2)$$

and an anti-Poisson map if

$$\{f,g\}_2 \circ F = -\{f \circ F, g \circ F\}_1, \ \forall f,g \in C^\infty(M_2)$$

The concept of Poisson map can be extended to the algebraic setup. Given two Poisson algebras $(\mathscr{A}_1, \{.,.\}_1)$, and $(\mathscr{A}_2, \{.,.\}_2)$ and a homomorphism of (abelian, associative) algebras, $\Phi : \mathscr{A}_2 \to \mathscr{A}_1$ we say that Φ is (anti-)Poisson if it is also a (anti-)homomorphism of Poisson algebras, i.e.

$$\Phi(\{f,g\}_2) = (-)\{\Phi(f), \Phi(g)\}_1.$$

In this paper we consider the case in which the Poisson algebras are subalgebras of the space of functions on certain manifolds and the homomorphism of algebras is induced by a map between the manifolds themselves.

3 Reduction of Poisson manifolds

Let C be a closed submanifold of (M,Γ). Can we define in a natural way a Poisson structure on C? The answer is negative, in general. What we can always achieve is to endow a certain subset of $C^\infty(C)$ with a Poisson algebra structure. The canonical procedure below follows in spirit reference [11], although we present some additional, new results.

We adopt the notation $\mathscr{A} = C^\infty(M)$ and take the ideal (with respect to the point-wise product of functions in $\mathscr{A}$. We will use the term Poisson ideal when we refer to an ideal with respect to the Poisson bracket).

$$\mathscr{I} = \{f \in \mathscr{A} \,|\, f(p) = 0, \ p \in C\}$$

Define $\mathscr{F} \subset \mathscr{A}$ as the set of *first-class functions*, also called the *normalizer* of $\mathscr{I}$,

$$\mathscr{F} = \{f \in \mathscr{A} \,|\, \{f, \mathscr{I}\} \subset \mathscr{I}\}.$$

Note that due to the Jacobi identity and the Leibniz rule $\mathscr{F}$ is a Poisson subalgebra of $\mathscr{A}$ and $\mathscr{F} \cap \mathscr{I}$ is a Poisson ideal of $\mathscr{F}$. Then, we have canonically defined a Poisson bracket in the quotient $\mathscr{F}/(\mathscr{F} \cap \mathscr{I})$. However, this is not what we want, as our problem was to find a Poisson bracket in $C^\infty(C) \cong \mathscr{A}/\mathscr{I}$ (or, at least, in a subset of it). To that end we define an injective map

$$\begin{array}{rccc} \phi: & \mathscr{F}/(\mathscr{F} \cap \mathscr{I}) & \longrightarrow & \mathscr{A}/\mathscr{I} \\ & f + \mathscr{F} \cap \mathscr{I} & \longmapsto & f + \mathscr{I} \end{array} \tag{1}$$

ϕ is an homomorpism of abelian, associative algebras with unit and then induces a Poisson algebra structure $\{.,.\}_C$ in the image, that will be denoted by $\mathscr{C}(\Gamma, M, C)$, i.e.:

$$\{f_1 + \mathscr{I}, f_2 + \mathscr{I}\}_C = \{f_1, f_2\} + \mathscr{I}. \qquad f_1, f_2 \in \mathscr{F}. \tag{2}$$

Remarks:

- *Poisson reduction* is a generalization of the symplectic reduction in the following sense:
  If the original Poisson structure is non-degenerate, it induces a symplectic structure ω in M. Then, we may canonically define on C the closed two-form $i^*\omega$, where $i : C \to M$ is the inclusion map. As described before, this presymplectic two-form in C defines a Poisson algebra for a certain subset of $C^\infty(C) \cong \mathscr{A}/\mathscr{I}$. The Poisson algebra obtained this way coincides with the one defined above.

- Note that the elements of $\mathscr{F} \cap \mathscr{I}$ are, in the language of physicists, the generators of *gauge transformations* or, in Dirac's terminology, the *first-class constraints*.

The problem is that in general ϕ is not onto and C cannot be made a Poisson manifold. The goal now is to use the geometric data of the original Poisson structure to interpret the algebraic obstructions.

Let N^*C (or $\mathrm{Ann}(TC)$) be the conormal bundle of C (or annihilator of TC) i.e., the subbundle of the pull-back $i^*(T^*M)$ consisting of covectors that kill all vectors in TC. Now one has the following

Theorem 1:
Assume that:
a) $\dim(\Gamma^\sharp_p(N^*_pC) + T_pC) = k,\ \forall p \in C$, *and*
b) $\Gamma^\sharp_p(N^*_pC) \cap T_pC = \{0\},\ \forall p \in C$
Then the map ϕ of (1) is an isomorphism of associative, commutative algebras with unit.

Proof: Condition b) implies that

$$T^*_pM = \mathrm{Ann}(\Gamma^\sharp_p(N^*_pC) \cap T_pC) = N^*_pC + \Gamma^{\sharp -1}_p(T_pC)$$

and then, $\Gamma^\sharp_p(T^*_pM) \subseteq \Gamma^\sharp_p(N^*_pC) + T_pC, \quad \forall p \in C$.

Now define a smooth bundle map:

$$\Upsilon : N^*C + TC \longrightarrow i^*TM$$

that maps $(\alpha_p, v_p) \in N^*_pC + T_pC$ to $\Gamma^\sharp_p\alpha_p + v_p$. Due to condition a) the map is of constant rank and then every smooth section of its image has a smooth preimage.

Take $f \in \mathscr{A}$. As shown before $\Gamma^\sharp_p(df)_p \in \Gamma^\sharp_p(N^*_pC) + T_pC$ for any $p \in C$. Then, the restriction to C of $\Gamma^\sharp df$ is a smooth section of the image of Υ. Let (α, v) be a smooth section of $N^*C + TC$ with $\Upsilon(\alpha, v)_p = \Gamma^\sharp_p(df)_p$ for $p \in C$. Now for any section α of N^*C there exists a function $g \in \mathscr{I}$ such that $\alpha_p = (dg)_p$ for any $p \in C$.

Hence, one has that $\tilde{f} = f - g \in \mathscr{F}$ and $\phi(\tilde{f} + \mathscr{F} \cap \mathscr{I}) = f + \mathscr{I}$. $\square$

When $\dim(\Gamma_p^\sharp(N_p^*C) + T_pC) = \dim(M)$ we can choose locally a basis $\{g_n\}$ of regular second-class constraints. The matrix of the Poisson brackets of the constraints $G_{mn} = \{g_m, g_n\}$ is invertible on C and the Poisson bracket of (2) is:

$$\{f + \mathscr{I}, f' + \mathscr{I}\}_C = \{f, f'\} - \sum_{m,n=1}^{\nu} \{f, g_n\} G_{nm}^{-1} \{g_m, f'\} + \mathscr{I} \tag{3}$$

which is the usual definition of the Dirac bracket restricted to C. In this case, of course, every function on C has a well-defined Poisson bracket and we get a Poisson structure on C.

Condition a) of Theorem 1 is not necessary as can be shown in the following

Example 1:

Take $M = sl(2)^*$. In coordinates (x_1, x_2, x_3) the linear Poisson bracket is given by $\{x_i, x_j\} = \varepsilon^{ijk} x_k$. Now define C by the constraints: $x_1 = 0, x_2 = 0$. Clearly,

$$\dim(\Gamma_p^\sharp(N_p^*C) + T_pC) = \begin{cases} 3 & \text{for } p \neq 0 \\ 1 & \text{for } p = 0 \end{cases}$$

and for any $f \in C^\infty(M)$ we may define $\tilde{f} = f - x_1 \partial_1 f - x_2 \partial_2 f \in \mathscr{F}$ such that $\phi(\tilde{f} + \mathscr{F} \cap \mathscr{I}) = f + \mathscr{I}$, i.e. ϕ is onto. The Poisson structure induced in this case is, of course, zero.

Condition b), however, is indeed necessary:

Theorem 2:

If map ϕ of (1) is onto then $\Gamma_p^\sharp(N_p^*C) \cap T_pC = \{0\}$

Proof: Assume that $\exists v_p \neq 0, v_p \in \Gamma_p^\sharp(N_p^*C) \cap T_pC$. It is enough to take a function $f \in \mathscr{A}$ such that its directional derivative at p in direction v_p does not vanish. Then $f + \mathscr{I}$ is not in the image of ϕ. $\square$

This result tells us that when $\Gamma_p^\sharp(N_p^*C) \cap T_pC \neq \{0\}$ one cannot endow C with a Poisson structure. The only functions on C that have got a well-defined Poisson bracket (i.e. the physical observables) are those in the image of ϕ. On the other hand, it is easy to see that all functions in the image of ϕ belong to the subalgebra of gauge invariant functions

$$\mathscr{A}_{inv} = \{f \in \mathscr{A} | \{f, \mathscr{F} \cap \mathscr{I}\} \subset \mathscr{I}\}.$$

One may wonder when the physical observables are precisely the gauge invariant functions. A sufficent condition is given by the following

Theorem 3:
If $\dim(\Gamma_p^\sharp(N_p^*C)+T_pC)=k,\ \forall p\in C$, *then* $\phi(\mathscr{F}/\mathscr{F}\cap\mathscr{I})=\mathscr{A}_{\mathrm{inv}}/\mathscr{I}$.

Before proving the theorem we will establish a Lemma that will be useful in the following.

Lemma 1:
The following two statements are equivalent:
a) $\dim(\Gamma_p^\sharp(N_p^*C)+T_pC)=k,\ \forall p\in C$
b) $\Gamma_p^{\sharp-1}(T_pC)\cap N^*C=\{(dg)_p|g\in\mathscr{F}\cap\mathscr{I}\},\ \forall p\in C.$

Proof:

a) $\Rightarrow$ b): Assume that $\dim(\Gamma_p^\sharp(N_p^*C)+T_pC)$ is constant on C. Then, $\mathrm{Ann}_p(\Gamma^\sharp(N^*C)+TC)=\Gamma_p^{\sharp-1}(T_pC)\cap N_p^*C$ is also of constant dimension and $\Gamma^{\sharp-1}(TC)\cap N^*C$ is a subbundle of N^*C whose fiber at every point of the base is spanned by a set of sections. For every section α of this subbundle there exists $g\in\mathscr{I}$ such that $\alpha_p=(dg)_p$. But since $(dg)_p\in\Gamma_p^{\sharp-1}(T_pC)$, it follows that $g\in\mathscr{F}\cap\mathscr{I}$.

The other inclusion is trivial as differential of first-class constraints are in N^*C (because they are constraints) and their hamiltonian vector fields transform constraints into constraints (because they are first-class) so their restrictions to C are in TC.

b) $\Rightarrow$ a): Assuming b) one has that $\dim(\Gamma_p^{\sharp-1}(T_pC)\cap N_p^*C)$ is a lower semicontinuous function on C because the fiber of $\Gamma_p^{\sharp-1}(T_pC)\cap N_p^*C$ at every point is spanned by local sections (see ref. [14]). For the same reason, $\dim(\Gamma_p^\sharp(N_p^*C)+T_pC)$ is also lower semicontinuous. But from the relation $\Gamma_p^\sharp(N_p^*C)+T_pC=\mathrm{Ann}_p(\Gamma^{\sharp-1}(TC)\cap N^*C)$ we infer that $\dim(\Gamma_p^\sharp(N_p^*C)+T_pC)$ is upper semicontinuous, so it is continuous and, being integer valued it is indeed constant. $\square$

Proof of Theorem 3: First note that $f\in\mathscr{A}_{inv}$ implies that $\Gamma_p^\sharp(df)_p\in\mathrm{Ann}_p(\{dg|g\in\mathscr{F}\cap\mathscr{I}\})$. But from the previous Lemma we have that the latter is equal to $\Gamma_p^\sharp(N_p^*C)+T_pC$.

Then, $\forall f\in\mathscr{A}_{inv}$ one has $\Gamma_p^\sharp(df)_p\in\Gamma_p^\sharp(N_p^*C)+T_pC$. And from here on the proof is like that of Theorem 1. $\square$

At first sight we might expect a result analogous to Theorem 2 for the case with gauge transformations in the constrained submanifold, namely that a necessary con-

dition for ϕ mapping onto the space of gauge invariant functions on C is that the space of hamiltonian vector fields of first-class constraints at every point coincides with $T_pC \cap \Gamma_p^\sharp(N^*C)$. This is not true, however, as shown by the following example in which the spaces above differ in some points whereas the image of map ϕ of (1) is $\mathscr{A}_{\rm inv}/\mathscr{I}$.

Example 2:

Take $M = \mathbb{R}^6 = \{(x_1, x_2, x_3, p_1, p_2, p_3)\}$ with the standard Poisson bracket $\{p_i, x_j\} = \delta_{ij}$. Now consider the constraints

$$g_i := p_i - x_i x_{\sigma(i)} \quad i = 1,2,3$$

with σ the cyclic permutation of $\{1,2,3\}$ s. t. $\sigma(1) = 2$. In this case

$$\dim(\Gamma_m^\sharp(N_m^*C) \cap T_mC) = \begin{cases} 1 & \text{for } m \neq 0 \\ 3 & \text{for } m = 0 \end{cases}$$

while the gauge transformations are restrictions to C of hamiltonian vector fields of fg with $f \in C^\infty(M)$ and $g = x_2 g_1 + x_3 g_2 + x_1 g_3$. It implies that at $m = 0$ the gauge transformations vanish and, hence, they do not fill $\Gamma_m^\sharp(N_m^*C) \cap T_mC$.

We will show that the image of map ϕ of (1) is $\mathscr{A}_{\rm inv}/\mathscr{I}$. In every class of $\mathscr{A}_{\rm inv}/\mathscr{I}$ we may take the only representative independent of the p_i's. Gauge invariant functions $f(x_1, x_2, x_3)$ are then characterized by:

$$(x_2 \partial_{x_1} + x_3 \partial_{x_2} + x_1 \partial_{x_3}) f = 0,$$

and for any of them we may define

$$\tilde{f} = f + \sum_i a_i g_i$$

with a_i smooth, given by

$$\begin{aligned} a_1(x_1,x_2,x_3) &= \frac{1}{x_1}[\partial_{x_2} f(x_1,x_2,x_3) - \partial_{x_2} f(0,x_2,x_3)] \\ a_2(x_1,x_2,x_3) &= \frac{1}{x_1}[\partial_{x_1} f(x_1,x_2,x_3) - \partial_{x_1} f(0,x_2,x_3)] \\ a_3(x_1,x_2,x_3) &= \frac{1}{x_2}\partial_{x_2} f(0,x_2,x_3) = -\frac{1}{x_3}\partial_{x_1} f(0,x_2,x_3) \end{aligned} \tag{4}$$

Now $\tilde{f}$ is first class and $\phi(\tilde{f} + \mathscr{F} \cap \mathscr{I}) = f + \mathscr{I}$. This shows that in this case the image of ϕ fills $\mathscr{A}_{\rm inv}/\mathscr{I}$.

In general, if $\dim(\Gamma_p^\sharp(N_p^*C) + T_pC)$ is not constant on C we cannot define a Poisson bracket even in the set of gauge invariant functions. The only thing we can assert

is that we have a Poisson algebra on the subset of $C^\infty(C)$ given by the image of ϕ. However, an efficient description of the functions in the image (the space of *observables*) is not available in the general case.

Remark:

C is said coisotropic if $\Gamma^\sharp(N^*C) \subseteq TC$. For such a C, $\mathscr{I} \subseteq \mathscr{F}$. Then, $\mathscr{F} \cap \mathscr{I} = \mathscr{I}$, $\mathscr{F} = \mathscr{A}_{inv}$ and ϕ is the identity map.

3.1 Poisson-Dirac submanifolds

In this subsection we would like to make contact between the results and terminology of this section in absence of gauge transformations and those appearing in two papers by Crainic and Fernandes [7] and Vaisman [15].

If $\Gamma^\sharp_p(N^*_pC) \cap T_pC = \{0\}$, $\forall p \in C$, C is called *pointwise Poisson-Dirac* in [7]. If, in addition, the induced Poisson bivector defined therein is smooth C is said a *Poisson-Dirac submanifold*. It is clear that ϕ onto implies that C is a Poisson-Dirac submanifold. The following is an example in which ϕ is not onto while C is still a Poisson-Dirac submanifold, being possible to endow it with a Poisson structure.

Example 3:

Consider $M = \mathbb{R}^4 = \{(x_1, x_2, p_1, p_2)\}$ with Poisson structure $\{p_i, x_j\} = \delta_{ij} x_i \exp(-1/x_i^2)$ smoothly extended to $x_i = 0$ and C defined by the constraints $g_1 = p_1 - x_2^2/2$, $g_2 = p_2 + x_1^2/2$. We can take $\sigma_i := x_i$ as coordinates on C.

$\Gamma^\sharp_p(N^*_pC) \cap T_pC = \{0\}$ on C but ϕ is not onto. For instance, take $f_i = x_i \in C^\infty(M)$. If we try to find a first-class function in the class $f_1 + \mathscr{I}$ (its pre-image by ϕ) we obtain for $x_i \neq 0$

$$\tilde{f}_1 := f_1 - \frac{x_1 \exp(-1/x_1^2)}{x_1^2 \exp(-1/x_1^2) + x_2^2 \exp(-1/x_2^2)}$$

which fails to extend continuously to $x_i = 0$. Then, f_1 does not belong to the image of ϕ. However, the hamiltonian vector field associated to this singular $\tilde{f}_1$ is smooth and we can define a Poisson structure on C:

$$\Gamma_C^{12}(\sigma_1, \sigma_2) = \{\tilde{f}_1, f_2\}(\sigma_1, \sigma_2, \frac{1}{2}\sigma_2^2, -\frac{1}{2}\sigma_1^2) = \frac{\sigma_1 \sigma_2}{\sigma_1^2 \exp(1/\sigma_2^2) + \sigma_2^2 \exp(1/\sigma_1^2)}$$

If ϕ is onto and, in addition, $\dim(\Gamma^\sharp_p(N^*_pC) + T_pC)$ is constant on C (i.e. the situation of Theorem 1), we have what is called in [7] a *constant rank Poisson-Dirac submanifold*.

Following [15], define a *normalization* of C by a *normal bundle* νC as a splitting $TM|_C = TC \oplus \nu C$. For every $p \in C$ there exists a neighborhood U where we can

choose adapted coordinates (g^A, y^α) such that, locally, $g^A|_{C\cap U} = 0$ and y^α are coordinates on $C\cap U$. Vaisman calls νC *algebraically* Γ*-compatible* if, in these coordinates, $\Gamma^{A\alpha}|_C = 0$. The relation with our map ϕ is given by the following

Theorem 4:
ϕ is onto iff there exists an algebraically Γ-compatible normal bundle.

Proof:
Only local properties in a neighborhood of each point of C matter for this proof.
$\Rightarrow$) Let (g^A, z^α) be local coordinates such that C is locally defined by $g^A = 0$ and z^α are coordinates on C. Take the pre-image by ϕ of the coordinate functions z^α and denote them by y^α. (g^A, y^α) are local coordinates such that $\Gamma^{A\alpha}|_C = 0$.
$\Leftarrow$) For any $f(g^A, y^\alpha) \in C^\infty(U)$, $\tilde{f}(g^A, y^\alpha) = f(0, y^\alpha) \in \mathscr{F}$ and $\tilde{f} - f \in \mathscr{I}$. Then, ϕ is onto. $\square$

4 Poisson-Sigma models

The Poisson-Sigma model is a two-dimensional topological Sigma model defined on a surface Σ and with a finite dimensional Poisson manifold (M,Γ) as target.

The fields of the model are given by a bundle map $(X,\psi) : T\Sigma \to T^*M$ consisting of a base map $X : \Sigma \to M$ and a 1-form ψ on Σ with values in the pullback by X of the cotangent bundle of M. The action functional has the form

$$S_{P\sigma}(X,\psi) = \int_\Sigma \langle dX, \wedge\psi\rangle - \frac{1}{2}\langle \Gamma\circ X, \psi\wedge\psi\rangle \tag{5}$$

where $\langle,\rangle$ denotes the pairing between vectors and covectors of M.

If X^i are local coordinates in M, σ^μ, $\mu = 1,2$ local coordinates in Σ, Γ^{ij} the components of the Poisson structure in these coordinates and $\psi_i = \psi_{i\mu} d\sigma^\mu$, the action reads

$$S_{P\sigma}(X,\psi) = \int_\Sigma dX^i \wedge \psi_i - \frac{1}{2}\Gamma^{ij}(X)\psi_i\wedge\psi_j \tag{6}$$

It is straightforward to work out the equations of motion in the bulk:

$$dX^i + \Gamma^{ij}(X)\psi_j = 0 \tag{7a}$$

$$d\psi_i + \frac{1}{2}\partial_i\Gamma^{jk}(X)\psi_j\wedge\psi_k = 0 \tag{7b}$$

One can show ([1]) that for solutions of (7a) the image of X lies within one of the symplectic leaves of the foliation of M.

Under the infinitesimal transformation

$$\delta_\varepsilon X^i = \Gamma^{ji}(X)\varepsilon_j \tag{8a}$$

$$\delta_\varepsilon \psi_i = d\varepsilon_i + \partial_i \Gamma^{jk}(X)\psi_j \varepsilon_k \tag{8b}$$

where $\varepsilon = \varepsilon_i dX^i$ is a section of $X^*(T^*(M))$, the action (6) transforms by a boundary term

$$\delta_\varepsilon S_{P\sigma} = -\int_\Sigma d(dX^i \varepsilon_i). \tag{9}$$

Formula (8) is not the most general transformation that leaves the action invariant up to a boundary term, but it gives a complete set of gauge transformations in the sense that any symmetry of the action is of type (8) up to terms vanishing on-shell (the so-called *trivial gauge transformations* of [9]). Then, it is not surprising that the commutator of two consecutive gauge tansformations of type (8) is not of the same form, i.e.

$$[\delta_\varepsilon, \delta_{\varepsilon'}]X^i = \delta_{[\varepsilon,\varepsilon']^*} X^i \tag{10a}$$

$$[\delta_\varepsilon, \delta_{\varepsilon'}]\psi_i = \delta_{[\varepsilon,\varepsilon']^*}\psi_i + \varepsilon_k \varepsilon'_l \partial_i \partial_j \Gamma^{kl}(dX^j + \Gamma^{js}(X)\psi_s) \tag{10b}$$

where $[\varepsilon,\varepsilon']^*_k := \partial_k \Gamma^{ij}(X)\varepsilon_i \varepsilon'_j$. Note that the term in parenthesis in (10b) is the equation of motion (7a) and then, as expected, it vanishes on-shell.

In this section we have analysed the equations of motion and gauge invariance in the bulk. In the following one we will address the subject of boundary conditions for the fields and how they affect the gauge transformations.

5 Boundary conditions

We study now the previous model on a surface with boundary and search for the boundary conditions (BC) which make the theory consistent.

In order to preserve the topological character of the theory one must choose the BC independent of the point of the boundary, as far as we move along one of its connected components. For the sake of clarity we will restrict ourselves in this section to one connected component (without mentioning it explicitly). In the next section we will discuss the relation between the BC in the possible different connected components of the boundary.

In surfaces with boundary a new term appears in the variation of the action under a change of X when performing the integration by parts:

$$\delta_X S = \int_{\partial\Sigma} \delta X^i \psi_i - \int_\Sigma \delta X^i (d\psi_i + \frac{1}{2}\partial_i \Gamma^{jk}(X)\psi_j \wedge \psi_k) \tag{11}$$

The BC must cancel the surface term.

Let us take the field

$$X|_{\partial\Sigma} : \partial\Sigma \to C \tag{12}$$

for an arbitrary (for the moment) closed embedded submanifold C of M (brane, in a more stringy language). Then $\delta X \in T_X C$ at every point of the boundary and the contraction of $\psi = \psi_i dX^i$ with vectors tangent to the boundary (that we will denote by $\psi_t = \psi_{it} dX^i$) must belong to $N^*_X(C)$ (the fiber over X of the conormal bundle of C).

On the other hand, by continuity, the equations of motion in the bulk must be satisfied also at the boundary. In particular,

$$\partial_t X = \Gamma^\sharp \psi_t$$

where by ∂_t we denote the derivative along the vector on Σ tangent to the boundary. As $\partial_t X$ belongs to $T_X C$ it follows that $\psi_t \in \Gamma^{\sharp -1}_X(T_X(C))$.

Both conditions for ψ_t imply that

$$\psi_t(m) \in \Gamma^{\sharp -1}_{X(m)}(T_{X(m)}C) \cap N^*_{X(m)}C, \text{ for any } m \in \partial\Sigma \tag{13}$$

which is the boundary condition we shall take for ψ_t.

We should check now that the BC are consistent with the gauge transformations (8).

In order to cancel the boundary term (9) $\varepsilon|_{\partial\Sigma}$ must be a smooth section of $N^*(C)$ and if (8) is to preserve the boundary condition of X, $\varepsilon|_{\partial\Sigma}$ must belong to $\Gamma^{\sharp -1}(TC)$. Hence,

$$\varepsilon(m) \in \Gamma^{\sharp -1}_{X(m)}(T_{X(m)}C) \cap N^*_{X(m)}C, \text{ for any } m \in \partial\Sigma \tag{14}$$

Next, we shall show that the the gauge transformations (8) with (14) also preserve (13). At this point we must restrict ourselves to the case in which

$$\dim(\Gamma^\sharp_p(N^*_p C) + T_p C) = k, \text{ for any } p \in C \tag{15}$$

In this case we can choose, at least locally, a set of regular constraints with a maximal number $(\dim(M) - \dim(\Gamma^\sharp_p(N^*_p C) + T_p C))$ of first-class ones. Let $\{\chi^a\}$ be the set of first-class constraints and $\{\gamma^A\}$ that of second-class ones. Local regularity means that for every point in C there is a neighborhood $U \subset C$ and a choice of constraints, such that differentials of the constraints at $p \in U$ span $N^*_p(C)$. U can be chosen so that we can also find coordinates $\{y^\alpha\}$ on C. Then $(y^\alpha, \chi^a, \gamma^A)$ form a set of local coordinates for an open subset of M containing U.

In these coordinates the Poisson structure satisfies:

$$\Gamma^{ab}|_C = 0, \quad \Gamma^{aA}|_C = 0, \quad \det(\Gamma^{AB})|_C \neq 0 \tag{16}$$

The boundary condition (13) translates in these coordinates into $\psi_t = \psi_{at} d\chi^a$. Hence, we must show that $\delta\psi_{\alpha t} = \delta\psi_{At} = 0$. Recalling (14) we also may write $\varepsilon|_C = \varepsilon_a d\chi^a$ and therefore,

$$\delta\psi_{\alpha t} = \partial_\alpha \Gamma^{ab}|_C \psi_{at} \varepsilon_b|_C$$

which vanishes because $\Gamma^{ab}|_C = 0 \Rightarrow \partial_\alpha \Gamma^{ab}|_C = 0$.

Showing that

$$\delta\psi_{At} = \partial_A \Gamma^{ab}|_C \psi_{at} \varepsilon_b|_C$$

also vanishes on C is more tricky, but it does, as a consequence of the Jacobi identity:

$$\begin{aligned} &\Gamma^{AB}\partial_A\Gamma^{ab} + \Gamma^{\alpha B}\partial_\alpha\Gamma^{ab} + \Gamma^{cB}\partial_c\Gamma^{ab} \\ &+\Gamma^{Ab}\partial_A\Gamma^{Ba} + \Gamma^{\alpha B}\partial_\alpha\Gamma^{Ba} + \Gamma^{cb}\partial_c\Gamma^{Ba} \\ &+\Gamma^{Aa}\partial_A\Gamma^{bB} + \Gamma^{\alpha a}\partial_\alpha\Gamma^{bB} + \Gamma^{ca}\partial_c\Gamma^{bB} \quad = \quad 0 \end{aligned} \tag{17}$$

Evaluating on C and using $\Gamma^{ab}|_C = \Gamma^{aA}|_C = 0$ and $\partial_\alpha\Gamma^{ab}|_C = \partial_\alpha\Gamma^{aA}|_C = 0$, one may check that all terms except the first one vanish. Then,

$$\Gamma^{AB}|_C \partial_A \Gamma^{ab}|_C = 0$$

Using now that $\Gamma^{AB}|_C$ is invertible, we conclude that $\partial_A\Gamma^{ab}|_C = 0$ and then $\delta\psi_{At} = 0$. A similar derivation proves that the gauge transformations close on-shell at the boundary (see (10)).

At this point one might want to weaken somehow the condition (15) to allow for more general BC. Firstly, we notice that some restriction must be imposed, as the existence of a maximal number of regular first-class constraints seems to be essential. Recall Example 2 of section 3 for a case in which gauge transformations do not preserve the BC of ψ_t. In this case the first class constraints with non-zero differential on C are generated by $x_2g_1 + x_3g_2 + x_1g_3$ which is not regular at 0.

A possible generalization of condition (15) is to assume that $\dim\{(dg)_p | g \in \mathscr{F} \cap \mathscr{I}\}$ is constant on C . With this assumption we may choose a maximal number of regular first-class constraints and the previous choice of coordinates works. In this case, however, $\det(\Gamma^{AB})|_C$ might be zero at some points, but only in the complement of an open dense set. An argument of continuity shows then that $\delta\psi_{At} = 0$ everywhere.

6 Hamiltonian analysis of the Poisson-Sigma model

We proceed to the hamiltonian study of the model with the BC of the previous section (in each connected component of the boundary) when $\Sigma = [0,\pi] \times \mathbb{R}$ (open string). The fields in the hamiltonian formalism are a smooth map $X : [0,\pi] \to M$ and a 1-form ψ on $[0,\pi]$ with values in the pull-back $X^*T^*(M)$; in coordinates, $\psi = \psi_{i\sigma} dX^i d\sigma$.

Consider the infinite dimensional manifold of smooth maps (X,ψ) with canonical symplectic structure Ω. The action of Ω on two vector fields (denoted for shortness δ,δ') reads

$$\Omega(\delta,\delta') = \int_0^\pi (\delta X^i \delta' \psi_{i\sigma} - \delta' X^i \delta \psi_{i\sigma}) d\sigma \tag{18}$$

The phase space $P(M;C_0,C_\pi)$ of the theory is defined by the constraint:

$$\partial_\sigma X^i + \Gamma^{ij}(X)\psi_{j\sigma} = 0 \tag{19}$$

and BC $X(0) \in C_0$ and $X(\pi) \in C_\pi$ for two closed submanifolds $C_u \subset M$, $u = 0,\pi$.

This geometry, with a boundary consisting of two connected components, raises the question of the relation between the BC at both ends. Note that due to eq. (19) X varies in $[0,\pi]$ inside a symplectic leaf of M. This implies that in order to have solutions the symplectic leaf must have non-empty intersection both with C_0 and C_π. In other words, only points of C_0 and C_π that belong to the same symplectic leaf lead to points of $P(M;C_0,C_\pi)$. In the following we will assume that this condition is met for every point of C_0 and C_π and correspondingly for the tangent spaces. That is, if we denote by J_0, J_π the maps

$$\begin{array}{rcl} J_0 : P(M,C_0,C_\pi) & \longrightarrow & C_0 \\ (X,\psi) & \longmapsto & X(0). \end{array} \tag{20}$$

and

$$\begin{array}{rcl} J_\pi : P(M,C_0,C_\pi) & \longrightarrow & C_\pi \\ (X,\psi) & \longmapsto & X(\pi). \end{array} \tag{21}$$

we assume that both maps are surjective submersions.

Vector fields tangent to the phase space satisfy the linearization of (19), i.e. $\delta\psi_{j\sigma}$ and δX^i are such that

$$\partial_\sigma \delta X^i = \partial_j \Gamma^{ki}(X)\psi_{k\sigma}\delta X^j + \Gamma^{ji}\delta\psi_{j\sigma} \tag{22}$$

with $\delta X(u) \in T_{X(u)} C_u$, $u = 0,\pi$.

The solution to the differential equation (22) is ([6])

$$\delta X^i(\sigma) = R^i_j(\sigma,0)\delta X^j(0) - \int_0^\sigma R^i_j(\sigma,\sigma')\Gamma^{jk}(X(\sigma'))\delta\psi_{k\sigma}(\sigma')d\sigma' \tag{23}$$

where R is given by the path-ordered integral

$$R(\sigma,\sigma') = \overleftarrow{P\exp[\int_{\sigma'}^\sigma A_\sigma(z)dz]}, \qquad A^i_j(z) = (\partial_j \Gamma^{ki})(X(z))\psi_{k\sigma}(z).$$

The canonical symplectic 2-form is only presymplectic when restricted to $P(M;C_0,C_\pi)$. The kernel is given by:

$$\begin{aligned}&\delta_\varepsilon X^i = \varepsilon_j \Gamma^{ji}(X)\\&\delta_\varepsilon \psi_i = d\varepsilon_i + \partial_i \Gamma^{jk}(X)\psi_j \varepsilon_k\end{aligned} \tag{24}$$

where ε, a section of $X^*(T^*M)$, is subject to the BC

$$\varepsilon(u) \in \Gamma^{\sharp-1}_{X(u)}(T_{X(u)}C_u) \cap N^*_{X(u)}(C_u), \quad \text{for } u = 0, \pi.$$

Note that a reparametrization of the path $\sigma \mapsto \sigma' = \sigma + \delta\sigma$ with $\delta\sigma(u) = 0, u = 0, \pi$ corresponds to a gauge symmetry with $\varepsilon_k = \psi_{k\sigma}\delta\sigma$. One may also check that as for the free BC or the coisotropic case the characteristic distribution of Ω has finite codimension.

As discussed in Section 2 the presymplectic structure induces a Poisson algebra $\mathscr{P}$ in the phase space $P(M,C_0,C_\pi)$. On the other hand, we have Poisson algebras in C_0 and C_π. We turn now to study the relation between them.

We first analyse under which circumstances a function $F(X,\psi) = f(X(0))$, $f \in C^\infty(M)$ belongs to $\mathscr{P}$, i.e. when it has a hamiltonian vector field δ_F. Solving the corresponding equation we see that the general solution is of the form (24) with

$$\varepsilon(0) - d f_{X(0)} \in N^*_{X(0)}(C_0), \qquad \varepsilon(0) \in \Gamma^{\sharp-1}_{X(0)}(T_{X(0)}C_0) \tag{25}$$

and

$$\varepsilon(\pi) \in \Gamma^{\sharp-1}_{X(\pi)}(T_{X(\pi)}C_\pi) \cap N^*_{X(\pi)}(C_\pi).$$

We saw in Section 3 (see Theorem 3) that assuming $\dim(\Gamma^\sharp(N^*_p(C_0)) + T_pC_0) = $ *const.*, equation (25) can be solved in $\varepsilon(0)$ if and only if F is a gauge invariant function (i.e. it is invariant under (24)). This is equivalent to saying that $f + \mathscr{I}_0$ belongs to the Poisson algebra $\mathscr{C}(\Gamma, M, C_0)$. (Here $\mathscr{I}_0$ is the ideal of functions that vanish on C_0).

Now, given two such functions F_1 and F_2 associated to $f_1 + \mathscr{I}_0, f_2 + \mathscr{I}_0 \in \mathscr{C}(\Gamma, M, C_0)$ and with gauge field ε_1 and ε_2 respectively, one immediately computes the Poisson bracket $\{F_1, F_2\}_P = \Omega(\delta_{F_1}, \delta_{F_2})$ to give

$$\{F_1, F_2\}_P = \Gamma^{ij}\varepsilon_{1i}(0)\varepsilon_{2j}(0) \tag{26}$$

This coincides with the restriction to C_0 of $\{f_1 + \mathscr{I}_0, f_2 + \mathscr{I}_0\}_{C_0}$ and defines a Poisson homomorphism between $\mathscr{C}(\Gamma, M, C_0)$ and the Poisson algebra of $P(M, C_0, C_\pi)$. This homomorphism is J_0^*, the pull-back defined by J_0, and the latter turns out to be a Poisson map. In an analogous way we may show that J_π is an anti-Poisson map and besides

$$\{f_0 \circ J_0, f_\pi \circ J_\pi\} = 0 \quad \text{for any } f_u \in \mathscr{C}(\Gamma, M, C_u),\ u = 0, \pi.$$

The previous considerations can be summarized in the following diagram

$$\mathscr{C}(\Gamma, M, C_0) \quad \overset{J_0^*}{\longrightarrow} \quad \mathscr{P} \quad \overset{J_\pi^*}{\longleftarrow} \quad \mathscr{C}(\Gamma, M, C_\pi) \tag{27}$$

in which J_0^* is a Poisson homomorphism, J_π^* antihomomorphism and the image of each map is the commutant (with respect to the Poisson bracket) of the other. In particular it implies that the reduced phase space is finite-dimensional, as claimed above.

This can be considered as a generalization of the symplectic dual pair to the context of Poisson algebras.

7 Conclusions

We have generalized the results of [5] to allow for non coisotropic branes in the Poisson-Sigma model.

In this more general situation we have to consider the reduction of Poisson brackets to a submanifold C of the original Poisson manifold M. This is achived in a canonical way at the price of ending up with a Poisson algebra on a subset of $C^\infty(C)$ rather than a Poisson structure on C. In more physical terms we can rephrase the previous considerations by saying that we are led to select certain observables on C which are the functions on the constrained phase space that belong to the Poisson algebra.

Two cases of interest are either when these observables fill $C^\infty(C)$ or when they are the functions invariant under the gauge transformations generated by the (first-class) constraints. We show that the constant rank of the Poisson bracket of (a local regular basis of) constraints is a sufficent conditions for having one case or the other. The Poisson bracket in these situations is the Dirac bracket (for gauge invariant functions in the second case). We show that in this setup it is possible to determine consistent BC for the Poisson-Sigma model with the base map at the boundary taking values in C. The resulting Poisson-Sigma model enjoys the basic properties of a topological theory (at the classical level), namely the characteristic distribution has finite codimension (the phase space reduced by the symmetries is finite-dimensional), and the reparametrization of the paths is among the gauge symmetries.

Quantization of the theory requires the introduction of ghost fields and BC for them. We have checked that in the constant rank case, BC for ghost fields can be chosen so that they are consistent with BRST symmetry at the boundary.

Feynman expansion of certain Green's functions of the Poisson-Sigma model on the disc, with free BC, gives rise to the formal deformation quantization of Poisson brackets found by Kontsevich [12]. It is natural to ask for the corresponding calculation with non-free BC. The coisotropic case has been worked out in [5] leading, under some suplementary assumptions, to a deformed associative product in the space of quantum-gauge invariant functions on C. In our case we would expect some similar

result but with the Dirac bracket playing the relevant role. However, this is a subtle issue: first note that in the Dirac bracket the inverse of the matrix of the Poisson brackets for second-class constraints appears. This inverse cannot be obtained by the standard perturbation theory of refs. [4],[5] around the zero Poisson bracket. The obstruction can be also seen from the fact that the propagator (in the standard perturbation theory) for the modes corresponding to the second-class constraints does not exist. A way out of this situation could be to redefine the perturbation theory by integrating out the fields ψ_A (with the notation of section 5, (16)) using the fact that Γ^{AB} is in this case invertible. The details of this computation will be the subject of further research.

In the case of a manifold $[0,\pi]\times\mathbb{R}$ with two connected components we analyse the relation between the BC at both components. We show that only points in the interesection of the two branes with a common symplectic leaf of M appear as boundary points in the classical solutions. If the evaluation of solutions at the boundary points J_0 and J_π are surjective submersion onto the corresponding branes C_0 and C_π, we may show that J_0 is a Poisson map while J_π is anti-Poisson. Furthermore, the pull-back by J_0^* of the Poisson algebra associated to C_0 is the commutant of the pull-back by J_π^* and viceversa. This defines a dual pair structure in the context of Poisson algebras that generalizes the concept of symplectic dual pair of [16].

When the target is a Poisson-Lie group ([8],[3]), the question of dual BC raises naturally. In ref. [3] it is shown that free boundary conditions are related by duality. A more general study was attempted in ([2]) in the coisotropic case. It would be interesting to address the problem in the context of more general BC.

The quantum counterpart of the strip was studied in [5] in the coisotropic case. There, the authors found a bimodule structure with the quantum algebras associated to C_0 (C_π) acting by the deformed product on the right (left) on the algebra associated to $C_0\cap C_\pi$. It is then natural to ask for the generalization of these results for the case of more general BC.

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